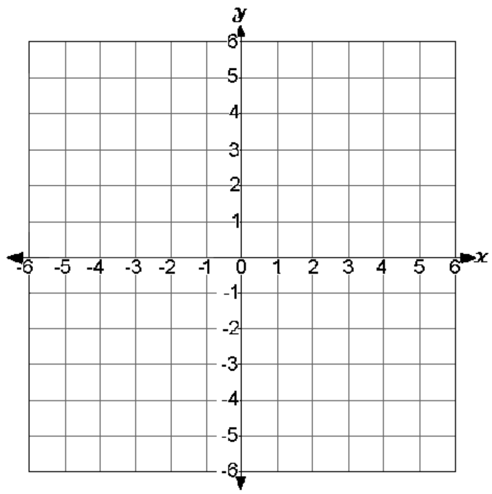
2022 03 26 workshop

Geometry



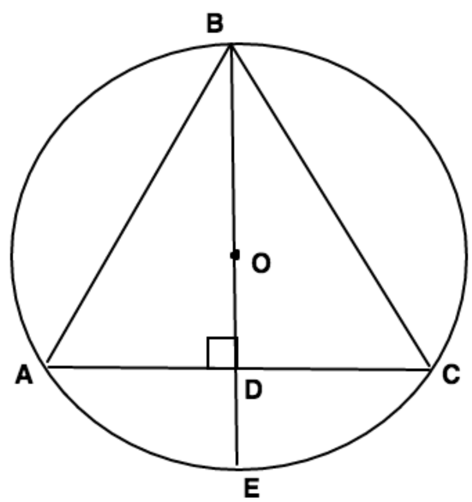
Terminology and definitions:
The figure above shows a 2 dimensional plain.
Point can be expressed as a pair (x, y) in the XY plain, where $-\infty < x, y < +\infty$.
The grid above shows the region $-6 \leq x, y \leq +6$.
x is defined as the $x-$ coordinate, and y is defined as the $y-$ coordinate of the point (x, y).
Specific points in the XY plain:
$x-$ axis: the infinitely long line of all points $(x, 0)$.
$y-$ axis: the infinitely long line of all points $(0, y)$.
Origin point: the point where $x = 0$ and $y = 0$, or $(0, 0)$.

Quiz 1: What is the measure of the distance, d, between the 2 grid points of $(3, -4)$ and $(-9, -9)$?

1. 10
2. 12
3. 13
4. 15
5. 17

To calculate the distance, d, on the grid, between two points (s, t) and (u, v), one can use Pythagorean Theorem:
$d^2 = (s - u)^2 + (t - v)^2$.
Thus, $d^2 = (3 - (-9))^2 + (-4 - (-9))^2 = 12^2 + 5^2$.
So, $d^2 = 144 + 25 = 169$. Thus, $d = 13$.

An equilateral triangle is a triangle whose all sides have the same measure. All angles of such triangle are the same: $60°$. Consider an equilateral triangle, ΔABC, circumscribed in a circle as in the figure below:
Line segment BD is a height of the triangle, so $\sphericalangle ADB = 90°$.



Quiz 2: What is the value (in °) of $\sphericalangle ABD$?
1. 15
2. 30
3. 45
4. 60
5. 90

Since all angles of the triangle are the same, $\measuredangle BAD = 60°$.
Given that for ΔABD, $\measuredangle ADB = 90°$, it follows that:
$\measuredangle ABD = 90° - 60° = 30°$.
Corollaries:
1. Triangles ΔABD and ΔCBD are congruent (i.e. they have the same measures of all their corresponding angles and sides).
2. The centre of the circle, O, is located on the line segment BD.
3. $AD = \frac{AC}{2}$.

Also, if the radius of the circle is r, then $OA = OB = r$, and based on corollary #3, it follows that $OD = \frac{OA}{2} = \frac{r}{2}$.

Quiz 3: If the radius of the circle is r, what is the value of AD in terms of r ?
1. $\frac{r}{3}$
2. $\frac{r}{2}$
3. $\frac{r\sqrt{3}}{3}$
4. $\frac{r\sqrt{3}}{2}$
5. r

OA is the radius of the circle so it satisfies: $AD^2 + OD^2 = OA^2$.
From earlier corollaries we know that $OD = \frac{r}{2}$.

Thus, $AD^2 = r^2 - \left(\frac{r}{2}\right)^2 = \frac{3}{4}r^2$.

$AD = \frac{r\sqrt{3}}{2}$.

Quiz 4: What is the area of an equilateral triangle circumscribed in a circle with area π ?

1. $3\sqrt{2}$
2. $\frac{3\sqrt{3}}{4}$
3. $\frac{3\sqrt{3}}{2}$
4. $\frac{\pi}{2}$
5. $\frac{\pi\sqrt{3}}{2}$

$\pi = \pi r^2$. So, $r = 1$. $AC = 2AD = \frac{2r\sqrt{3}}{2} = \sqrt{3}$.

$AC = \sqrt{3}$ is a side of the triangle.

$OB = r = 1$, $OD = \frac{OB}{2} = \frac{1}{2}$.

So, $BD = OB + OD = 1 + \frac{1}{2} = \frac{3}{2} = h$.

The area of the triangle is $\frac{AC \times h}{2} = \frac{\sqrt{3}}{2} \times \frac{3}{2} = \frac{3\sqrt{3}}{4}$.

Combinatorics, permutations, and combinations.
Consider a set of $N > 0$ members: $\{a_1, a_2, a_3, \cdots, a_N\}$.
Let us start with a simple case where $N = 4$, (a set of 4 members).
Suppose that we have a group of 4 students: $\{A, B, C, D\}$.

Quiz 5: If we want to put these 4 students in a line, in how many ways can we do it?

1. 4
2. 10
3. 20
4. 24
5. 28

Suppose that student A is at the first position in the line. There are 3 ways to select a student for the second position and 2 ways to select a student for the third position. So, there are $6 = 3 \times 2$ ways to select positions of the other three students (as shown below):
$\{A, B, C, D\}$,$\{A, B, D, C\}$,$\{A, C, B, D\}$,$\{A, C, D, B\}$,$\{A, D, B, C\}$, and $\{A, D, C, B\}$. Thus, similarly, there are $6 = 3 \times 2 \times 1$ ways if A is in any of the 4 positions (first, second, third, fourth). So, the total number to put 4 students in a line is $4 \times 3 \times 2 \times 1 = 24$. Or, for the more general case, the number is:
$N! = N \times (N - 1) \times (N - 2) \times \cdots \times 1$.

Quiz 6: In the case that $N = 5$, in how many ways one can select a certain student to be in first position in the line, (position 1), and another certain student to be in position 2?
1. 10
2. 20
3. 30
4. 60
5. 120

There are 5 ways to select a certain student to be in position 1, and for each of these ways, there are 4 ways to select one of the remaining 4 students to be in position 2. So in total, the number is $5 \times 4 = \frac{5\times4\times3\times2\times1}{3\times2\times1} = \frac{5!}{3!} = 20$.
Or, in the general case, $\frac{N!}{(N-2)!}$.

Quiz 7: In how many ways can one select two certain students of the above 5 students?
1. 3
2. 5
3. 8
4. 10
5. 15

We can restate the question as if we want to select 2 certain students to be in positions 1 and 2, but regardless of order. So, we have to divide the answer of the previous Quiz by $2 = 2 \times 1 = 2!$. So the answer is: $\frac{5!}{3!\times2!} = 10$.
Or, in general: $\frac{N!}{(N-2)!\times2!}$.

Notation: The number of ways to select K elements of a group of N elements is $C(N, K)$.
Thus: $C(N, K) = \frac{N!}{(N-K)! \times K!}$.

Quiz 8: In how many ways can you select 3 students out of a total of 11 students?

1. 165
2. 330
3. 360
4. 495
5. 540

Plug in the numbers, $C(11, 3) = \frac{11!}{(11-3)! \times 3!} = \frac{11!}{8! \times 3!}$. calculating: $\frac{11!}{8! \times 3!} = \frac{9 \times 10 \times 11}{1 \times 2 \times 3} = 3 \times 5 \times 11 = 165$.

Quiz 9: In how many ways can you select a group of 3 students of the above group of 11 students, if you want that either student A, or student B, or student C is a member of this subgroup?

1. 28
2. 36
3. 45
4. 56
5. 109

Consider all options (but not counting any option more than once):
1. Student A + 2 of the 10 remaining students.
2. Student B + 2 of the 9 remaining students (other than A).
3. Student C + 2 of the 8 remaining students (other than A, B).
1. $\boldsymbol{C(10,2) = \frac{10!}{8!\times 2!} = \frac{9\times 10}{2} = 45}$ **.**
2. $\boldsymbol{C(9,2) = \frac{9!}{7!\times 2!} = \frac{8\times 9}{2} = 36}$ **.**
3. $\boldsymbol{C(8,2) = \frac{8!}{6!\times 2!} = \frac{7\times 8}{2} = 28}$ **.**
So, total number of options based on the above condition is:
$\boldsymbol{45 + 36 + 28 = 109}$ **.**
Also, note that the number of options if none of A, B, or C are selected is: $\boldsymbol{C(8,3) = \frac{8!}{5!\times 3!} = \frac{6\times 7\times 8}{1\times 2\times 3} = 7\times 8 = 56}$ **.**
And note that: $\boldsymbol{56 + 109 = 165}$ **.**

Toss 5 coins.
What is the total number of all various sequences of Heads and Tails?
For each of the coins we can get either Head or Tail, or a total of 2 different options. So the number of sequences is $\boldsymbol{2^5 = 32}$ **.**
An example for such is the sequence HHTTH, and the probability of any such sequence is $\boldsymbol{\frac{1}{32}}$ **.**

Quiz 10: What is the probability to get exactly 3 Heads in 5 tosses.

1. $\boldsymbol{\frac{1}{16}}$
2. $\boldsymbol{\frac{1}{8}}$
3. $\boldsymbol{\frac{1}{4}}$
4. $\boldsymbol{\frac{5}{16}}$
5. $\boldsymbol{\frac{3}{8}}$

The number of options to select 3 individual tosses with Heads out of the 5 tosses is the same as the number of ways to select 3 students out of 5 students.
Thus, the number is: $C(5,3) = \frac{5!}{3!\times 2!} = \frac{4\times 5}{2} = 10$ **.**
The total number of different sequences is 32.
Thus, the probability is $\frac{10}{32} = \frac{5}{32}$ **.**

Throw 2 dice.
Define an outcome of a throw as a pair (x, y) **where** x **is the number shown on the first die and** y **is the number shown on the second die.**

Quiz 11: What is the total number of different pairs (x, y) **?**

1. 12
2. 15
3. 21
4. 24
5. 36

There are 6 options for x, and 6 options for y. So the total number of different pairs (x, y) is $6 \times 6 = 36$.

Quiz 12:
What is the probability that $x + y = 8$?

1. $\frac{1}{18}$
2. $\frac{1}{9}$
3. $\frac{5}{36}$
4. $\frac{3}{18}$
5. $\frac{7}{36}$

$$8 = 2 + 6 = 3 + 5 = 4 + 4 = 5 + 3 = 6 + 2 .$$

So, there are a total of 5 pairs out of 36 pairs whose sum is 8.
So, $P(sum = 8) = \frac{5}{36}$.

Conditional probability.
The notation of conditional of probability is $P(A|B)$, where A and B are events with some probability each. The notation $P(A|B)$ is defined as the probability that event A happened given the fact that we know that event B happened.

To clarify, consider the following case when rolling a single die.
Example 1: What is $P(A|B)$ for the two probability events below?
Event A: "The die displays the number 1".
Event B: "The die displays a number less than the number 5".
Event B includes 4 out of the six possible events, each with the same probability of $\frac{1}{6}$.
So, knowing that B happened, the probability that A happened is $P(A|B) = \frac{1}{4}$.
Example 2: What is $P(B|A)$ given the two events above?
Event A is that the die displays the number 1. Thus, the die certainly displays a number which is less than 5. Thus $P(B|A) = 1$.

Now, consider a more complicated conditional probability case.
Quiz 13: Throw 1 die. Record the number that was thrown. Keep throwing the die until the sum of the recorded numbers is greater than 1. What is the probability that sum is 3?

1. $\frac{1}{9}$
2. $\frac{5}{36}$
3. $\frac{1}{6}$
4. $\frac{7}{36}$
5. $\frac{7}{18}$

First, identify what are events A and B. Event A: "Keep throwing the die until the sum of the recorded numbers is greater than 1". Event B: "The sum of all throws is 3".
A single throw occurs if the recorded number is one of the following: 2, 3, 4, 5, or 6. Each has a probability of $\frac{1}{6}$, so the probability of a single throw is $\frac{5}{6}$.
The probability of a total of 2 throws is $\frac{1}{6}$, and the probability of more than 2 throws is 0. So, the probability of event A is in fact 1.
At probability of $\frac{1}{6}$ we get a throw of 3, "the sum of 3 in one throw". If 2 throws are needed, then the only sequence to produce "sum of 3" is the sequence $(1, 2)$. Its probability is $\frac{1}{36}$.
So, the total probability for "sum of 3" is $P(B) = \frac{1}{6} + \frac{1}{36} = \frac{7}{36}$.
Thus, $P(B|A) = \frac{7}{36}$.
Note that this example was easy to do because $P(A) = 1$.

Quiz 14: GIVEN the same throwing condition, we know that "sum of 3" was achieved. What is the probability that the first throw was the number 1?

1. $\frac{1}{36}$
2. $\frac{1}{18}$
3. $\frac{1}{14}$
4. $\frac{1}{9}$
5. $\frac{1}{7}$

To clarify, consider all sequences (x, y) and treat the sequences $(3, y)$ as they are 6 different sequences leading to "sum of 3" out of a total of the 36 different sequences of 2 throws. Another sequence that leads to "sum of 3" is the sequence $(1, 2)$. So there are 7 sequences of "sum of 3" of which only one sequence satisfies "the first throw was 1". So, the probability that the first throw was 1 under this condition is $\frac{1}{7}$.

Mixtures measurements and units

There are 2 bags of construction mix. Bag A weighs $20kg$, of which 45% is sand (by weight) and the rest is pebbles. You pour this bag into a container and add another bag B of mixture of sand and pebbles. The weight of the combined mixture is $50kg$ with 50% of it sand (by weight).

Quiz 15: What percentage of pebbles was in bag B? Round the answer to the nearest whole number.

1. 42
2. 43
3. 46
4. 47
5. 48

Total weight of sand in bag A: $9kg$ (45% of 20).
Total weight of sand in combined mixture: $25kg$ (50% of 50).
Weight of Bag B: $30kg$ ($50 - 20$).
Weight of sand in bag B: $16kg$ ($25 - 9$).
Weight of pebbles in bag B: $14kg$ ($30 - 16$).
Percentage of pebbles in bag B: $\frac{14}{30} \times 100 = \frac{140}{3} = 46.666\cdots$.
Round to the nearest whole number: 47

More Geometry
Acute angle: angle of less than $\mathbf{90°}$.
Right angle: angle of $\mathbf{90°}$.
Obtuse angle: angle of a triangle between $\mathbf{90°}$ deg and $\mathbf{180°}$.
Convex polygon: a polygon whose all angles are less than $\mathbf{180°}$.

Sum of all angles of a polygon with $\boldsymbol{N}$ sides.
Polygon with $\boldsymbol{N}$ sides can be divided into $\boldsymbol{N-2}$ Triangles.
So sum of angles is $(\boldsymbol{N-2}) \times \mathbf{180°}$

Regular polygon: a polygon whose all angles and all sides are the same.
Examples are equilateral triangle and square.

Congruent triangles: two triangles are congruent if all their corresponding sides have the same values.

Similar triangles: two triangles are similar if all their corresponding angles have the same values.

Quiz 16: What is the value of each angle of a regular pentagon?

Quiz 17:
How many non congruent triangles can be formed using entire sides and/or entire lengths of diagonals of a regular hexagon?